Proceedings of the American Academy of Arts and Sciences.

VOL. XXXVII. NO. 5. — SEPTEMBER, 1901.

ON RULED LOCI IN n-FOLD SPACE.

BY HALCOTT C. MORENO.

ON RULED LOCI IN n-FOLD SPACE.

By Halcott C. Moreno.

Presented by W. E. Story, May 8, 1901. Received June 1, 1901.

The present paper is a discussion of those loci in n-fold space that can be generated by flats whose equations involve a single arbitrary parameter. The ruled loci of space of three dimensions can be represented in this way.

I. Loci derived from an $(n-1)$-flat whose Equation involves a Single Arbitrary Parameter; Developables.

1. *Description of the derived loci.*

Let us consider the loci derived from the equation

$$A = 0,$$

the equation of an $(n-1)$-flat involving a single arbitrary parameter λ. If the parameter enters rationally, we suppose it to enter to as high a degree as n, the number of ways of the space. If the parameter enters rationally to the degree m where $m < n$, the locus is of a special kind to be discussed later. As the parameter varies continuously we have a 1-fold infinite system of $(n-1)$-flats.

Two consecutive $(n-1)$-flats of the system intersect in an $(n-2)$-flat whose equations are

$$A = 0, \qquad \frac{\partial A}{\partial \lambda} = 0.$$

If from these equations we eliminate the parameter there remains a single equation of an $(n-1)$-spread, S_{n-1}, which is ruled by the 1-fold infinite system of $(n-2)$-flats.

Three consecutive $(n-1)$-flats of the system intersect in an $(n-3)$-flat whose equations are

$$A = 0, \qquad \frac{\partial A}{\partial \lambda} = 0, \qquad \frac{\partial^2 A}{\partial \lambda^2} = 0.$$

These $(n-3)$-flats may be considered as arising from the intersection of two consecutive $(n-2)$-flats of the system of $(n-2)$-flats. The elimination of the parameter from these equations gives a restricted system equivalent to two independent equations. The system represents an $(n-2)$-spread, S_{n-2}, which is ruled by the $(n-3)$-flats.

In like manner r consecutive $(n-1)$-flats of the system intersect in an $(n-r)$-flat whose equations are

$$A=0,\ \frac{\partial A}{\partial \lambda}=0,\ \ldots\ \frac{\partial^{r-2} A}{\partial \lambda^{r-1}}=0.$$

Any of these $(n-r)$-flats may be considered as arising from the intersection of two consecutive $(n-r+1)$-flats of the system of $(n-r+1)$-flats that are the intersections of $r-1$ consecutive $(n-1)$-flats of the system. The elimination of the parameter from these equations gives a restricted system equivalent to $r-1$ independent equations. These equations represent an $(n-r+1)$-spread, S_{n-r+1}, which is ruled by the 1-fold infinite system of $(n-r)$-flats.

The locus of the intersections of n consecutive $(n-1)$-flats of the system is a curve, while $n+1$ consecutive $(n-1)$-flats do not in general have any common intersection.

We will use S_k to denote that one of the related spreads of this system that is of k ways. It is geometrically evident that each one of these spreads is a developable spread.*

Considered in this light we see that the $(n-2)$-spread is a double spread on S_{n-1} corresponding to the cuspidal edge or edge of regression in ordinary threefold space.†

The S_{n-3} is a double spread on S_{n-2}, etc., and S_1 on S_2. We see also that S_{n-3} is a triple spread on S_{n-1}; Killing calls it doubly stationary. Finally, S_1 is an $(n-1)$-tuple curve on S_{n-1}; it is a multiple curve on all the other spreads of the system.‡

If the equation

$$A=0$$

contains k arbitrary parameters connected by $k-1$ equations

$$\phi=0,\ \chi=0,\ \ldots\ \psi=0,$$

* Killing, Nicht-Euklidische Raumformen, p. 195 et seq.

† Puchta calls the S_{n-1} the most general developable spread in n-fold space. Puchta, Ueber die allgemeinsten abwickelbaren Räume, ein Beitrag zur mehrdimensionalen Geometrie. Wien. Berichte, CI.

‡ Killing, loc. cit.

we can, theoretically, solve these equations for $k - 1$ of the parameters in terms of the remaining one, so that this case is the same as the previous one.

The actual elimination may be avoided. Let the parameters be λ, μ, ... ν. Differentiate totally all the equations,

$$\frac{\partial A}{\partial \lambda} d\lambda + \frac{\partial A}{\partial \mu} d\mu + \ldots\ldots + \frac{\partial A}{\partial \nu} d\nu = 0$$

$$\frac{\partial \phi}{\partial \lambda} d\lambda + \frac{\partial \phi}{\partial \mu} d\mu + \ldots\ldots + \frac{\partial \phi}{\partial \nu} d\nu = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

From these we may eliminate the differentials,

$$B \equiv \begin{Vmatrix} \frac{\partial A}{\partial \lambda}, & \frac{\partial A}{\partial \mu}, & \ldots\ldots & \frac{\partial A}{\partial \nu} \\ \frac{\partial \phi}{\partial \lambda}, & \frac{\partial \phi}{\partial \mu}, & \ldots\ldots & \frac{\partial \psi}{\partial \nu} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi}{\partial \lambda}, & \frac{\partial \psi}{\partial \mu}, & \ldots\ldots & \frac{\partial \psi}{\partial \nu} \end{Vmatrix} = 0$$

This is the equation of an $(n - 1)$-flat. The equation involves k parameters but they are connected by $k - 1$ equations. Two consecutive $(n - 1)$-flats of the system intersect in an $(n - 2)$-flat whose equations are $A = 0$, $B = 0$.

Three consecutive $(n - 1)$-flats of the system intersect in the $(n - 3)$-flat,

$$A = 0, \quad B = 0, \quad C = 0,$$

where C is the determinant B, with A replaced by B. The equation of the S_{n-1} is found by eliminating the parameters between the equations of the $(n - 2)$-flats and the equations connecting the parameters. The equations of the other spreads are derived in a similar manner. The system of related spreads is of the same character as before.

2. *Mutual relations of connected loci.*

Let us consider more in detail these connected loci. We will use F_k to denote a k-flat of the 1-fold infinite system of k-flats. Two consecu-

tive F_{n-1}'s intersect in an F_{n-2}, three in an F_{n-3}, r in an F_{n-r}, $n-2$ in an F_2 or plane, $n-1$ in an F_1 or line, n in an F_0 or point. There is a 1-fold infinite system of these F_{n-2}'s which are generators of S_{n-1}, a 1-fold infinite system of F_{n-3}'s, generators of S_{n-2}, a 1-fold infinite system of lines generators of S_2, the developable surface.

Through any F_{n-2} there pass two consecutive F_{n-1}'s, through any F_{n-3} there pass three consecutive F_{n-1}'s, through any F_0, n consecutive F_{n-1}'s. Through any F_{n-3} there pass two consecutive F_{n-2}'s, through any F_{n-4} there pass two consecutive F_{n-3}'s and three consecutive F_{n-2}'s, and so on.

We may then reverse this process and start with the curve of the system. Through any two consecutive points of the curve there passes a line, an F_1, through any three consecutive points an osculating plane, an F_2, through any four consecutive points an osculating 3-flat, an F_3, through any n — consecutive points an osculating $(n-1)$-flat, an F_{n-1}.*

That these operations may give unique results this curve must lie in the n-fold space and in no flat space of a less number of ways. If the curve lie in a k-flat, where $k \overline{\overline{<}} n-1$, all the k-flats through $k+1$ consecutive points coincide and definite $(k+1)$-flats are not determined at all. By a theorem of Clifford, such a curve must be of an order as great as n.†

This theorem has been generalized by Veronese.‡

Let us consider any curve in n-fold space whose equations are,

$$\phi = 0,\ \chi = 0, \ldots.\ \psi = 0,$$

a restricted system equivalent to $n-1$ independent equations. The equations of the tangent at any point P' of this curve are linear equations whose coefficients are functions of the n non-homogeneous coordinates, $x', y', \ldots. v'$. The same thing is true of the equations of any of the osculating flats at the point P'. The osculating $(n-1)$-flat is given by a single equation, the coefficients of which are functions of these n quantities $x', y', \ldots v'$. If we regard these as n parameters they are connected by the equations,

$$\phi' = 0,\ \chi' = 0, \ldots.\ \psi' = 0,\S$$

* We shall say a k-flat osculates a curve if it contains $k+1$ consecutive points of it. Killing, loc. cit.

† Clifford, Classification of Loci; Mathematical Papers, pp. 305–331.

‡ Veronese, Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens, Mathematische Annalen XIX.

§ $\phi' \equiv \phi\ (x', y', \ldots v')$, etc.

a restricted system equivalent to $n-1$ independent equations. We have then the case of an $(n-1)$-flat whose equation involves n parameters connected by $n-1$ independent relations; this is equivalent to the case of a single equation containing one arbitrary parameter. We may, in general, consider the system of developables as given by an $(n-1)$-flat whose equation contains a single arbitrary parameter or k parameters connected by $k-1$ equations.*

3. *The tangent* $(n-1)$*-flats that are common to* $n-1$ $(n-1)$*-spreads envelop a developable.*

The equation in homogeneous coördinates of any $(n-1)$-flat may be written

$$x = \alpha y + \beta z + \ldots\ldots + \gamma w.$$

This equation involves n independent parameters; if we connect them by any $n-1$ independent equations we shall have the equation of an $(n-1)$-flat that contains but a single independent parameter, so that the 1-fold infinite system of $(n-1)$-flats represented by it envelop a developable. The tangent $(n-1)$-flat at any non-singular point of a developable S_{n-1} contains the generating F_{n-2} through that point and touches the S_{n-1} all over this flat.† We may speak of this developable S_{n-1} as enveloped by its tangent F_{n-1}'s. If then we impose on an arbitrary $(n-1)$-flat any conditions that give rise to $n-1$ independent equations between the coefficients in its equation, the $(n-1)$-flat will envelop a developable S_{n-1}.

Let
$$U = 0$$

be the equation of an $(n-1)$-spread. The equation of the tangent $(n-1)$-flat at any ordinary point P' is

$$x\frac{\partial U'}{\partial x'} + y\frac{\partial U'}{\partial y'} + \ldots\ldots + w\frac{\partial U'}{\partial w'} = 0.$$

If we impose on the equation of the arbitrary $(n-1)$-flat the conditions that it shall be this tangent $(n-1)$-flat, the coefficients in the two equations must be proportional. We must have then

$$\frac{\frac{\partial U'}{\partial x'}}{-1} = \frac{\frac{\partial U'}{\partial y'}}{\alpha} = \ldots\ldots \frac{\frac{\partial U'}{\partial w'}}{\gamma}.$$

From these equations by means of the equation

$$U' = 0,$$

* Salmon, Geometry of Three Dimensions, p. 286. † Killing, loc. cit.

we may eliminate the coördinates of P' leaving a single equation in $\alpha, \beta, \ldots \gamma$. For an $(n-1)$-flat to be tangent to an $(n-1)$-spread, one relation between the coefficients that enter into their equations must be satisfied. We conclude then that the $(n-1)$-flats that touch $n-1$ $(n-1)$-spreads envelop an S_{n-1}.

Let us consider only those tangent $(n-1)$-flats to an $(n-1)$-spread that touch it at the point of an $(n-2)$-spread that lies on it. Let

$$U = 0$$

be the equation of the $(n-1)$-spread and let

$$U = 0, \; V = 0, \ldots,$$

a restricted system equivalent to two independent equations, be the equations of the $(n-2)$-spread on it. We derive now the equations

$$\frac{\frac{\partial U'}{\partial x'}}{-1} = \frac{\frac{\partial U'}{\partial y'}}{\alpha} = \ldots \frac{\frac{\partial U'}{\partial w'}}{\gamma}$$

and

$$U' = 0, \; V' = 0, \ldots$$

If we eliminate the parameters from these equations there remains a restricted system equivalent to two independent equations in the coefficients $\alpha, \beta, \ldots \gamma$. For an $(n-1)$-flat to be tangent to an $(n-1)$-spread at a point of an $(n-2)$-spread on it requires two conditions between the coefficients in the equation of the $(n-1)$-flat. These two conditions may be used as part of the $n-1$ conditions that connect the coefficients of an $(n-1)$-flat that envelops a developable S_{n-1}. We have then the theorem that the $(n-1)$-flats that are tangent to ρ $(n-1)$-spreads at the points of ρ $(n-2)$-spreads that lie one on each $(n-1)$-spread, and are tangent to σ other $(n-1)$-flats, where $n-1 = 2\rho + \sigma$, envelop a developable.

In a similar manner for an $(n-1)$-flat to be tangent to an $(n-1)$-spread at a point of an $(n-3)$-spread that lies on it imposes three conditions on the coefficients that enter into the equation of the $(n-1)$-flat. To be tangent to the $(n-1)$-flat at a point of an $(n-4)$-flat on it requires four conditions, etc. To be tangent to an $(n-1)$-spread at a point of a curve that lies on it requires $n-1$ conditions between the coefficients, which is just sufficient to make the $(n-1)$-flat envelop a developable.

We have then the general theorem that the $(n-1)$-flats that are tangent to ρ $(n-1)$-spreads at points of ρ $(n-k)$-spreads that lie one

on each, tangent to σ $(n-1)$-spreads at points of σ $(n-k+1)$-spreads that lie one on each, tangent to τ $(n-1)$-spreads at points of τ $(n-2)$-spreads that lie one on each, and finally tangent to v other $(n-1)$-spreads, where $\rho, \sigma, \ldots \tau, v$, are non-negative integers connected by the relation

$$n-1 = k.\rho + (k-1)\,\sigma + \ldots . + 2\,\tau + v,$$

envelop a developable S_{n-1}.

Similar cases occur in three-fold space where we have the tangent planes that are common to two surfaces enveloping a developable surface as do the tangent planes to a surface at the points of a curve on that surface.*

4. *Some additional properties of developables; sections.*

Other properties of an S_{n-1} may be deduced by regarding it as the envelope of an $(n-1)$-flat whose equation involves a single parameter.† Through any point in space can be drawn a definite number of tangent F_{n-1}'s to the S_{n-1}. For substitute the coördinates of the point in the equation of the variable $(n-1)$-flat and there is a certain finite number of values of the parameter that satisfy the equation.

Any F_{n-1} of the system meets its consecutive F_{n-1} in a definite F_{n-2}, a generator of S_{n-1} whose equations are,

$$A = 0, \frac{\partial A}{\partial \lambda} = 0.$$

Any three consecutive F_{n-1}'s meet in a definite F_{n-3}, a generator of S_{n-2}, whose equations are,

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \frac{\partial^2 A}{\partial \lambda^2} = 0.$$

Any $n-1$ consecutive F_{n-1}'s meet in a definite line F_1, a generator of S_2, whose equations are,

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \ldots \frac{\partial^{n-2} A}{\partial \lambda^{n-2}} = 0.$$

Finally, any n consecutive F_{n-1}'s meet in a definite point of the curve of regression of S_2. The equations of the F_0 are,

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \ldots \frac{\partial^{n-1} A}{\partial \lambda^{n-1}} = 0.$$

---

* Salmon, Geometry of Three Dimensions, p. 547.

† Salmon, Geometry of Three Dimensions, p. 289 et seq.

In general $n+1$ consecutive F_{n-1}'s do not have any common intersection, for the $n+1$ equations,

$$A=0, \frac{\partial A}{\partial \lambda}=0, \ldots \frac{\partial^n A}{\partial \lambda^n}=0,$$

have no common solutions. If we regard these equations as homogeneous in the $n+1$ coördinates we may form their resultant, and the values of the parameter that cause this determinant to vanish, give special points where $n+1$ consecutive F_{n-1}'s intersect. These points are cusps on the curve S_1.

Reciprocally there will, in general, be a finite number of F_{n-1}'s that go through $n+1$ consecutive points of S_1.

Veronese has shown that a curve in n-fold space has $3n$ singularities which are connected by $3(n-1)$ relations, an extension of the Pluecker-Cayleyan characteristics of a twisted curve in three-fold space.*

In this we have assumed that the variables that enter into the equation of the enveloping $(n-1)$-flat cannot be expressed in terms of fewer than $n+1$ independent linear functions of the variables alone. If they could be expressed in terms of ν such linear functions, where $\nu \lesseqgtr n$, the developable S_{n-1} is a conoid with an $(n-\nu)$-way head, a case to be considered later.

The developable S_k of the series is ruled by $(k-1)$-flats, F_{k-1}'s. The S_k, where $2 \lesseqgtr k \lesseqgtr n-1$ can be given by means of its enveloping F_k whose equations involve a single parameter. The $n-k$ equations of the F_k must however be of the form

$$A=0, \frac{\partial A}{\partial \lambda}=0, \ldots, \frac{\partial^{n-k-1} A}{\partial \lambda^{n-k-1}}=0,$$

as we have previously seen. Even the S_1 may be represented in this manner.

Any $(n-1)$-flat $\qquad B=0$

cuts the S_{n-1} in a developable $(n-2)$-spread, for it cuts the system of F_{n-1}'s in a system of $(n-2)$-flats that intersect consecutively in $(n-3)$-flats. We may see this in another way. By means of this new equation we can eliminate one variable from the equation of the enveloping $(n-1)$-flat. The resulting equation in n variables may evidently be considered as the envelope of an $(n-2)$-spread in a new $(n-1)$-fold space. The $(n-1)$-flat cuts any S_k of the system in a $(k-1)$-way

* Veronese, loc. cit.; Killing, loc. cit. p. 197 et seq.

developable. In general any r-flat where $r \gtreqqless n - k + 1$ cuts any S_k in a developable $(k + r - n)$-spread.

Any F_{n-1} of the system cuts the S_{n-1} in an $(n - 2)$-spread, and the F_{n-2} that it has in common with the consecutive F_{n-1} appears twice in the intersection, so that the proper $(n - 2)$-spread is of order less by two than the order of S_{n-1}. This $(n - 2)$-spread is also a developable.

An F_{n-2} is met by the consecutive F_{n-2} in an F_{n-3}; it is met by any other F_{n-2} in an $(n - 4)$-flat. In general, where $4 \lesseqgtr n$, there are a 2-fold infinite system of these $(n - 4)$-flats and their locus is an $(n - 2)$-spread which is a double spread on S_{n-1}. In the case of cones and conoids this double spread may be of fewer than $n - 2$ ways. Thus in four-fold space the planes which join a line to the successive points of an irreducible conic form a three-way developable. This developable is a conoid and the one-way head is the only multiple locus on the conoid. In three-fold space cones are the only developable surfaces that do not possess a proper double curve, if we call the cuspidal curve a double curve. In general there is a double curve distinct from the cuspidal curve. We will assume that we have the general case of a developable and not a cone or conoid. The total double spread on S_{n-1} consists in general of two parts, S_{n-2} and Σ_{n-2}, where Σ_{n-2} is the locus of the 2-fold infinite system of $(n - 4)$-flats arising from the intersection of non-consecutive F_{n-2}'s, while S_{n-2} is the locus of the 1-fold infinite system of $(n - 3)$-flats arising from the intersection of consecutive F_{n-2}'s.

Any three non-consecutive F_{n-2}'s intersect in an $(n - 6)$-flat; there are in general a 3-fold infinite system of such $(n - 6)$-flats whose locus is an $(n - 3)$-spread, a triple spread on S_{n-2}. Any $(n - 6)$-flat is the intersection of three $(n - 4)$-flats of Σ_{n-2} and any such $(n - 4)$-flat contains a 1-fold infinite system of such $(n - 6)$-flats. This 1-fold infinite system of $(n - 6)$-flats does not, in general, fill out the $(n - 4)$-flat, for this would require a 1-fold infinite system of them. The total triple spread on S_{n-1} consists in general of two parts S_{n-3} and Σ_{n-3} where Σ_{n-3} is the locus of the 3-fold infinite system of $(n - 6)$-flats. We can supply a similar mode of reasoning to the spreads of higher multiplicities on S_{n-1}. The spreads $S_{n-2}, S_{n-3}, \ldots$ are developable, but $\Sigma_{n-2}, \Sigma_{n-3}, \ldots$ are not developable.

5. *Special case where the parameter enters rationally.*

Let us illustrate this theory by the case of the developable which is the envelope of the $(n - 1)$-flat,

$$a\,t^m + m\,b\,t^{m-1} + \tfrac{1}{2}\,m\,(m - 1)\,c\,t^{m-2} + \ldots . = 0,$$

where t is a variable parameter, $a, b, c, \ldots$ are linear functions of the coördinates that are not expressible in terms of any ν linear functions of the coördinates where $\nu \lesseqgtr n$, and m is an integer which is not less than n, the number of ways of the space. Two consecutive F_{n-1}'s intersect in the F_{n-2},

$$a\,t^{m-1} + (m-1)\,b\,t^{m-2} + \frac{(m-1)\,(m-2)}{2\,!}\,c\,t^{m-3} + \ldots . + e = 0,$$

$$b\,t^{m-1} + (m-1)\,c\,t^{m-2} + \ldots + e\,t + f = 0.$$

The elimination of the parameter from these equations gives the equation of S_{n-1}. The result is the discriminant of the original equation placed equal to zero; the order of S_{n-1} is then $2\,(m-1)$.*

Three consecutive F_{n-1}'s intersect in the F_{n-3},

$$a\,t^{m-2} + (m-2)\,b\,t^{m-3} + \ldots . = 0,$$

$$b\,t^{m-2} + (m-2)\,c\,t^{m-3} + \ldots . + e = 0,$$

$$c\,t^{m-2} + \ldots . + e\,t + f = 0.$$

The equations of S_{n-2} are found by eliminating the parameter from these equations. The result is a restricted system equivalent to two independent equations; the order of the system, i. e., the order of S_{n-2} is $3\,(m-2)$.†

Similarly k consecutive F_{n-1}'s intersect in the F_k, given by the k equations,

$$a\,t^{m-k+1} + (m-k+1)\,b\,t^{m-k} + \ldots . = 0$$

$$b\,t^{m-k+1} + (m-k+1)\,c\,t^{m-k} + \ldots . = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots + e\,t + f = 0.$$

The elimination of the parameter from these equations gives a restricted system equivalent to $k-1$ independent equations, the equations of S_{n-k+1}. The order of S_{n-k+1} is seen to be $(k+1)\,(m-k)$.

Lastly the intersection of n consecutive F_{n-1}'s is the point, F_0, given by the equations,

$$a\,t^{m-n+1} + (m-n+1)\,b\,t^{m-n} + \ldots . = 0$$

$$b\,t^{m-n+1} + (m-n+1)\,c\,t^{m-n} + \ldots . = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots + e\,t + f = 0.$$

---

* Salmon, Higher Algebra, art. 105.

† This is the condition that the three equations have a common root; Salmon, Higher Algebra, art. 277.

The elimination of the parameter from these equations gives a restricted system equivalent to $n-1$ independent equations, the equation of S_1 whose order is $n\,(m-n+1)$.

We can find the equations of those exceptional points where $n+1$ consecutive F_{n-1}'s intersect in a point, if we eliminate the parameter from the $n+1$ equations

$$\begin{aligned} a\,t^{m-n} + (m-n)\,b^{m-n-1} + \dots &= 0 \\ b\,t^{m-n} + (m-n)\,c^{m-n-1} + \dots &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ \dots\dots\dots\dots\dots + e\,t + f &= 0. \end{aligned}$$

The result is a restricted system equivalent to n independent equations; it is of order $(n+1)\,(m-n)$, which is the number of such points, cusps on S_1. We may verify this result by forming the resultant of these $(n+1)$ equations. If we eliminate the variables from these equations we have a determinant of order $n+1$. If we expand this result t enters to the degree $(n+1)\,(m-n)$ so that there are $(n+1)\,(m-n)$ values of t that cause this resultant to vanish. These values of t give the special points in question.*

Any double point on S_{n-1} must lie on two F_{n-2}'s. We may find the equations of the total double spread on S_{n-1}, by expressing the conditions that the equations of an F_{n-2} regarded as equations in the parameter, have two roots in common. These conditions are †

$$\text{(I)} \qquad \left\| \begin{array}{llll} a, & (m-1)\,b, & \dfrac{(m-1)\,(m-2)}{2\,!}\,c, & \dots\dots\dots \\ & a, & (m-1) & b, \dots\dots\dots \\ & & & \dots\dots\dots \\ & & & \dots\dots\dots\dots\dots e \\ & b, & (m-1) & c, \dots\dots\dots \\ & & & b, \dots\dots\dots \\ & & & \dots\dots\dots \\ & & & \dots\dots (m-1)\,e, f \end{array} \right\| = 0,$$

---

* For $n=3$, these results agree with those of Salmon, Geometry of Three Dimensions, p. 296. Neither the results there nor these hold when the system has stationary $(n-1)$-flats.

† Salmon, Higher Algebra, art. 275.

where there are $2\ (m-2)$ rows and $2\,m-3$ columns. This restricted system is of order $\frac{1}{2}\ (2\,m-3)\ (2\,m-4)$. The double spread represented by these equations consists of two distinct parts, S_{n-2} and Σ_{n-2}. The order of Σ_{n-2} must be,

$$\tfrac{1}{2}\ (2\,m-3)\ (2\,m-4)-3\ (m-2)=2\ (m-2)\ (m-3).$$

A triple point on S_{n-1} must lie on three F_{n-2}'s. We may find the equations of the total triple spread on S_{n-1} by expressing the conditions that the equations of the F_{n-2} have three common roots. These conditions are expressed by means of a rectangular system similar in form to (I), in which however there are only $2\ (m-3)$ rows and $2\,m-4$ columns. The order of the restricted system is

$$\frac{1}{3\,!}\ (2\,m-4)\ (2\,m-5)\ (2\,m-6).$$

This triple spread consists of two distinct parts, S_{n-3} and Σ_{n-3}. The order of Σ_{n-3} must be

$$\frac{1}{3\,!}(2m-4)(2m-5)(2m-6)-4(m-3)=\frac{2}{3}(m-3)(m-4)(2m-1).$$

In like manner we can find the equations of the total k-tuple spread on S_{n-1}, by expressing the conditions that the equations of the F_{n-1} have k roots in common. These conditions are expressed by means of a rectangular system similar to (I), in which, however, there are only $2\ (m-k)$ rows and $2\,m-k-1$ columns. This is a restricted system equivalent to k independent equations, of order $\frac{1}{k\,!}\ (2\,m-k-1)$ $(2\,m-k-2)\ \ldots\ (2\,m-2\,k)$. This spread consists of two parts, S_{n-k} and Σ_{n-k}; the order of the latter is

$$\frac{1}{k\,!}\ (2\,m-k-1)\ (2\,m-k-2)\ \ldots\ (2\,m-2\,k)-(k+1)\ (m-k).$$

The total $(n-1)$-tuple curve on S_{n-1} is given by means of a restricted system similar to (I), in which, however, there are only $2\ (m-n+1)$ rows and $2\,m-n$ columns. We have then a restricted system equivalent to $n-1$ independent equations whose order is

$$\frac{1}{(n-1)\,!}\ (2\,m-n)\ (2\,m-n-1)\ \ldots\ (2\,m-2\,n+2).$$

The order of the curve Σ is,

$$\frac{1}{(n-1)!}(2m-n)(2m-n-1)\ldots.(2m-2n+2)-n(m-n+1).*$$

The equations of all the n-tuple points on S_{n-1} are given by means of a rectangular system similar to (I), in which, however, there are only $2(m-n)$ rows and $2m-n-1$ columns. They form a restricted system equivalent to n independent equations, whose order is

$$\frac{1}{n!}(2m-n-1)(2m-n-2)\ldots.(2m-2n);$$

this is the number of n-tuple points. The number of the n-tuple points other than the cusps on S_1, are

$$\frac{1}{n!}(2m-n-1)(2m-n-2)\ldots.(2m-2n)-(n+1)(m-n).$$

These points necessarily lie on Σ_1; they are either n-tuple points on Σ_1, or else they are n-tuple points on the combined curves S_1 and Σ_1. In three-fold space the double curve on the developable may have tripl points on it; it can have no double points off of the cuspidal curve.

If $m=n$, then the order of S_1 is n, and there are no cuspidal points on the curve; this is the rational normal curve of Veronese.† The order of S_{n-1} in this case is $2(n-1)$; no developable S_{n-1} can be of lower order unless it is a cone or conoid, for no curve of lower order than n can lie in the n-fold space without at the same time lying in a space of fewer than n ways.

Let us consider the case where $m=p<n$, where p is an integer. Any $p+1$ consecutive F_{n-1}'s intersect in an F_{n-p-1} whose equations are

$$A=0, \frac{\partial A}{\partial t}=0, \ldots \frac{\partial^p A}{\partial t^p}=0.$$

If we use two homogeneous parameters λ and μ instead of the single parameter t, these equations may be written

---

* For $n=3$, this result agrees with that in Salmon, Geometry of Three Dimensions, p. 296.

† Veronese, loc. cit.

$$\frac{\partial^p A}{\partial \lambda^p} = 0, \frac{\partial^p A}{\partial \lambda^{p-1} \partial \mu} = 0, \ldots \frac{\partial^p A}{\partial \mu^p} = 0,$$

in which form the parameter no longer appears. Any $p + 1$ consecutive F_{n-1}'s intersect in the same F_{n-p-1} as any other consecutive $p + 1$; i. e., all the F_{n-1}'s of the system contain the same F_{n-p-1}. Any p-flat that does not meet this F_{n-p-1} cuts S_{n-1} in a developable $(p - 1)$-spread of order $2(p - 1)$. This developable $(p - 1)$-spread of order $2(p - 1)$ lying in a p-flat is exactly similar to the case in n-fold space where $m = n$. The curve at the base of this system is of order p; it is the rational normal curve of p-fold space. Hence we may derive this system by joining by lines all points of a developable $(p - 1)$-spread of order $2(p - 1)$ in a p-fold space, to all points of an $(n - p - 1)$-flat that does not meet the p-flat that contains the $(p - 1)$-spread. S_{n-1} is a conoid of $(n - 2)$-flats with an $(n - p - 1)$-way head. The generating F_{n-2}'s of S_{n-1} arise from the junction of the $(n - p - 1)$-way head with the generating $(p - 2)$-flats of the $(p - 1)$-spread. The generating F_{n-3}'s of S_{n-2} arise from the junction of the $(n - p - 1)$-way head with the system of generating $(p - 3)$-flats of the $(p - 2)$-spread, and so on. Any conoid ruled by a 1-fold infinite system of q-flats with a $(q - 1)$-way head is a developable spread, but not so if it has only an r-way head where $r \overline{\gtrless} q - 2$. The latter spread is a developable only when the consecutive q-flats have $(q - 1)$-way intersection. Any conoid ruled by a 1-fold infinite system of $(n - 2)$-flats that have an $(n - 3)$-flat in common is a developable, but if they have only an $(n - k)$-flat in common where $k \overline{\gtrless} 4$, the conoid may or may not be developable. The cones and conoids with a 2-fold infinite system of generators are not developables at all.

The points of intersection of two consecutive generators are not in general points of intersection of three generators. The equations of a generator may be written

$$e + (m - 1)\, d + \frac{(m - 1)\,(m - 2)}{2\,!}\, c + \ldots . = 0,$$

$$f + (m - 1)\, e + \frac{(m - 1)\,(m - 2)}{2\,!}\, d + \ldots . = 0.$$

The points of intersection of three generators of the system are given by the equations

$$\left\| \begin{array}{cccc} e, (m-1)\,d, & \dfrac{(m-1)\,(m-2)}{2} & c, \ldots . \\ e, & (m-1) & d, \ldots . \\ & & \ldots\ldots \\ f, (m-1)\,e, & \dfrac{(m-1)\,(m-2)}{2} & d, \ldots . \\ f, & (m-1) & e, \ldots . \\ & & \ldots\ldots \end{array} \right\| = 0.$$

where there are $2\,(m-2)$ rows and $2\,m-2$ columns.

For $t = 0$ we have the particular $(n-2)$-flat

$$e = 0, \; f = 0.$$

The next consecutive generator has for its equations,

$$\begin{aligned} e + \delta t \,.\, d &= 0 \\ f + \delta t \,.\, e &= 0 \end{aligned}$$

The intersection of the two consecutive generators is the $(n-3)$-flat whose equations are

$$e = 0, \; f = 0, \; d = 0.$$

This F_{n-3} does not generally lie on the total triple spread for one of the equations of that system, namely

$$\left| \begin{array}{l} \dfrac{(m-1)\;(m-2)\,c}{2}, \ldots\ldots \\ \qquad\quad (m-1)\,d, \ldots\ldots \\ \qquad\qquad\qquad e, \ldots\ldots \\ \qquad\qquad\qquad \ldots\ldots\ldots \\ (m-1)\;\;(m-2)\,d, \ldots\ldots \\ \qquad\quad (m-1)\,e, \ldots\ldots \\ \qquad\qquad\qquad f, \ldots\ldots \\ \qquad\qquad\qquad \ldots\ldots\ldots \end{array} \right| = 0,$$

is not generally satisfied when the equations of the F_{n-3} are satisfied.

The points that satisfy both these systems of equations are evidently points on two consecutive generators and at the same time points on three generators.

If there is a linear relation between f, e, and d, then these two consecutive generators intersect in an $(n-2)$-flat, *i. e.*, they are coincident and we have a stationary generator of the system. If

$$e \equiv 0,$$

then

$$f = 0$$

is the equation of a stationary generator of the system. The equation of the developable S_{n-1} in this case is

$$\left| \begin{array}{cccc} f, & 0, & \frac{(m-1)\ (m-2)}{2!} & d, \ldots\ldots \\ & f, & & 0, \ldots\ldots \\ & & & f, \ldots\ldots \\ & & & \ldots\ldots\ldots \\ 0, & d, & \frac{(m-1)\ (m-2)}{2!} & c, \ldots\ldots \\ & 0, & & d, \ldots\ldots \\ & & & 0, \ldots\ldots \\ & & & \ldots\ldots\ldots \end{array} \right| = 0.$$

We see that f is a factor of the left member of this equation. When this factor is thrown out, the residual or proper developable is of a degree less by one than before. The orders of the multiple loci previously given are also reduced, they only holding when there are no stationary F_{n-1}'s in the system. By means of Veronese's formulae we see that when there are β stationary F_{n-1}'s the order of the λ-way developable is reduced from $(m-\lambda+1)(m-n+\lambda)$ to $(n-\lambda+1)(m-n+\lambda)-(n-\lambda)\beta$.

6. *Tangent flats to a* p*-spread where* $2 \overline{\gtrless}$ p.

a. *Definitions.*

We have treated up to this point the various developables that arise from a curve in n-fold space. We shall show now that similar developables do not arise from the consideration of the tangent flats of spreads of more than one way.

Let

$$U = 0$$

be the equation of an $(n-1)$ spread of order m. We shall use the points (1), (2), $\lambda\,(1) + \mu\,(2)$ to denote the points whose coördinates are $x_1, y_1, \ldots w_1, x_2, y_2, \ldots w_2$, and $\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2, \ldots, \lambda w_1 + \mu w_2$, respectively, so that $\lambda\,(1) + \mu\,(2)$ represents a point on the line (12), i. e., the line joining (1) and (2). We denote the result of substituting the coördinates of the points (1) or (2) in U by U_1, and U_2 respectively. We use the symbols

$$\Delta_2\, U_1 \equiv \left(x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_1} + \ldots + w_2 \frac{\partial}{\partial w_1}\right) U_1,$$

$$\Delta_2\, U \equiv \left(x_2 \frac{\partial}{\partial x} + y_2 \frac{\partial}{\partial y} + \ldots + w_2 \frac{\partial}{\partial w}\right) U,$$

$$\Delta\, U_2 \equiv \left(x \frac{\partial}{\partial x_2} + y \frac{\partial}{\partial y_2} + \ldots + w \frac{\partial}{\partial w_2}\right) U_2,$$

$$\Delta^k_2\, U_1 \equiv \left(x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + \ldots + w_2 \frac{\partial}{\partial w_1}\right)^k U_1.$$

In the last case the operator is to be applied k times to U_1. Now $\lambda\,(1) + \mu\,(2)$ is a point on the line (12), if it is also a point of the $(n-1)$-spread, it must satisfy the equation of the spread. Substitute the coördinates of $\lambda\,(1) + \mu\,(2)$ in U and we have

$$\lambda^m\, U_1 + \lambda^{m-1} \mu\, \Delta_2\, U_1 + \frac{\lambda^{n-2} \mu^2}{2\,!} \Delta^2_2\, U_1 + \ldots$$

$$\ldots + \frac{\mu^m}{m\,!} \Delta_2^m\, U_1 = 0.$$

The m values of $\lambda : \mu$ that satisfy this equation determine the m points where the line (12) meets the $(n-1)$-spread. If the point (1) lies on the spread then

$$U_1 = 0.$$

If the line (12) meets the spread twice at the point (1), then

$$U_1 = 0, \;\; \Delta_2\, U_1 = 0.$$

The equation of the locus of all the lines that meet the spread twice at (1) is

$$\Delta\, U_1 = 0.$$

From the analogy of three-fold space, this locus of lines is called the tangent $(n - 1)$-flat to the $(n - 1)$-spread, at the point (1).* At each point of an $(n - 1)$-spread there is in general a unique tangent $(n - 1)$-flat.

A p-spread is given by the equations,

$$U = 0,$$
$$V = 0,$$
$$W = 0,$$
$$\cdots\cdot$$

a restricted system equivalent to $n - p$ independent equations. In a similar manner the equations of the locus of all lines that meet the p-spread twice at any non-singular point (1) are,

$$\Delta\, U_1 = 0,$$
$$\Delta\, V_1 = 0,$$
$$\Delta\, W_1 = 0,$$
$$\cdots\cdots$$

Since these equations are linear we may select any $n - p$ that are independent and the rest are superfluous.† We have then a p-flat which from analogy is called the tangent p-flat to the p-spread at the point (1). At any point of a p-spread there is in general a unique tangent p-flat.‡

We define a tangent r-flat at a given point of the p-spread where $r < p$ as an r-flat that lies in the tangent p-flat at that point and contains the point. If $r > p$, we define a tangent r-flat at a given point as an r-flat that contains the tangent p-flat at that point. The locus of tangent lines then to a p-spread is simply the locus of tangent p-flats to the spread. The locus of tangent planes, 3-flats, . . . , $(p - 1)$-flats is this same locus. If then there are developables that arise from a p-spread, where $1 < p$ their number is not so great as $n - p - 1$, for

* This proof is given in Dr. Story's Lectures on Hyperspace.

† Some of these equations may be satisfied identically; this will be the case when (1) is a multiple point on any of the $(n - 1)$-spreads, but not a multiple point on the p-spread.

‡ Dr. Story, Lectures on Hyperspace.

the tangent lines, tangent planes, tangent 3-flats, ..., tangent p-flats all have the same locus. The planes through two consecutive lines, the 3-flats through two consecutive planes, etc., the p-flats through two consecutive $(p-1)$-flats all have this same locus possibly of a certain multiplicity.

b. *Intersections of consecutive tangent flats.*

We shall show further that $(p+1)$-flats cannot in general be passed through two consecutive tangent p-flats, for such p-flats do not in general have $(p-1)$-flats in common. Tangent p-flats at consecutive points of a p-spread where $1 \gtreqless p \gtreqless \frac{n}{2}$ do intersect in points at least. Let

$$U = 0,$$
$$V = 0,$$
$$\cdots\cdot$$

a restricted system equivalent to $n - p$ independent equations be the equations of the p-spread. Let

$$P' \equiv (x', y', \ldots) \text{ and } P'' \equiv (x' + d\,x', y' + d\,y', \ldots)$$

be consecutive points of the spread. The tangent p-flats at these points are

$$\Delta U' \equiv x \frac{\partial U'}{\partial x'} + y \frac{\partial U'}{\partial y'} + \ldots . = 0,$$

$$\Delta V' \equiv x \frac{\partial V'}{\partial x'} + y \frac{\partial V'}{\partial y'} + \ldots . = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and

$$\Delta U'' \equiv \Delta U' + x \left(\frac{\partial^2 U'}{\partial x'^2} d\,x' + \frac{\partial^2 U'}{\partial x' \partial y'} d\,y' + \ldots . \right) = 0,$$

$$\Delta V'' = \Delta V' + x \left(\frac{\partial^2 V'}{\partial x'^2} d\,x' + \frac{\partial^2 V'}{\partial x' \partial y'} d\,y' + \ldots . \right) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

All of these equations being linear, only $n - p$ equations in each set can be independent. In general, $2\,(n - p)$ equations for such a value of p have no common intersection. In the present case the resultant of any $n + 1$ equations of the combined systems vanishes for any consecutive points P' and P'' on the p-spread, so that no more than n equations of the combined systems can be independent. Hence tangent p-flats at con-

secutive points of a p-spread intersect in a point at least. Tangent planes at consecutive points of a surface in n-fold space intersect at least in points. These tangent planes do not generally intersect in lines unless the surface lies in a space of three ways. Let us take p to represent the tangent plane at any point P of the surface and take p', p'', p''', . . . to represent the tangent planes at the points P', P'', P''', . . . consecutive points of an infinitesimal closed curve about P. If p and p' intersect in a line they determine a three-flat. If the consecutive tangent planes intersect in lines, then p'' has a line in common with both p and p' and so p'' lies in this three-flat. In a similar manner it can be shown that p', p'', p''' . . . , all the tangent planes consecutive, to p lie in the same three-flat with it, i. e. a unique three-flat is determined at each point of the surface that contains the tangent plane at the point and all the tangent planes consecutive to it. Since however this three-flat is determined by any two of these tangent planes, the three-flats corresponding to P and P' any two consecutive points are the same. Take now any curve through P that lies on the surface. Since the three-flats corresponding to any two consecutive points of the curve are the same, it follows that the three-flats corresponding to all the points of this curve are the same. If we take a different curve through P the same thing is true of the points of it. The three-flats corresponding to all the points of these two curves are the same since they are all the same as the three-flat corresponding to P. From this it follows that the whole surface and all of its tangent planes lie in the same three-flat. Hence if in general all the tangent planes consecutive to any tangent plane of a surface lie in the same three-flat with it, then the whole surface lies in this three-flat.

In the same way it may be shown that if in general all the tangent planes consecutive to the tangent plane at any point of a surface lie in the same four-flat with it that the whole surface lies in this four-flat. Hence in n-fold space not only do the consecutive tangent planes of a surface not intersect in lines, but all the tangent planes consecutive to any tangent plane do not lie in the same four-flat with it.

c. *The locus of the intersections of the tangent plane at any point of a surface with the consecutive tangent planes.*

In a four-fold space let the surface be given by

$$U = 0,$$
$$V = 0,$$
$$\dots\,,$$

a restricted system equivalent to two independent equations. The tangent planes at P' and P'', any two consecutive points, have for their equations

$$\Delta U' = x \frac{\partial U'}{\partial x'} + y \frac{\partial U'}{\partial y'} + \dots = 0,$$

$$\Delta V' = x \frac{\partial V'}{\partial x'} + y \frac{\partial V'}{\partial y'} + \dots = 0,$$

$$\dots\dots\dots\dots\dots\dots$$

and

$$\Delta U'' = \Delta U' + x \left(\frac{\partial^2 U'}{\partial x'^2} d x' + \frac{\partial^2 U'}{\partial x' \partial y'} d y' + \dots\right) + \dots = 0,$$

$$\Delta V'' = \Delta V' + x \left(\frac{\partial^2 V'}{\partial x'^2} d x' + \frac{\partial^2 V'}{\partial x' \partial y'} d y' + \dots\right) + \dots = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Let us take the first two equations in each set to be independent, then the rest are superfluous. Since P' and P'' are points of the surface,

$$U' = 0$$

$$V' = 0$$

$$U'' = U' + \frac{\partial U'}{\partial x'} d x' + \dots = 0,$$

$$V'' = V' + \frac{\partial V'}{\partial x'} d x' + \dots = 0,$$

From these three sets of equations we derive

$$x \left(\frac{\partial^2 U'}{\partial x'^2} d x' + \frac{\partial^2 U'}{\partial x' \partial y'} d y' + \dots\right) + \dots = 0,$$

$$x \left(\frac{d^2 V'}{\partial x'^2} d x' + \frac{\partial^2 V'}{\partial x' \partial y'} d y' + \dots\right) + \dots = 0,$$

$$\frac{\partial U'}{\partial x'} d x' + \dots = 0,$$

$$\frac{\partial V'}{\partial x'} d x' + \dots = 0.$$

These four equations are homogeneous in the five differentials $d x'$, $d y', \dots$ We may take one of these differentials to be zero and eliminate the other four. We have

$$\left| \begin{array}{llll} x \frac{\partial^2 U'}{\partial x'^2} + y \frac{\partial^2 U'}{\partial x' \partial y'} + \ldots, & x \frac{\partial^2 U'}{\partial x' \partial y'} + y \frac{\partial^2 U'}{\partial y'^2} + \ldots, & -, & -, \\ x \frac{\partial^2 V'}{\partial^2 x'} + y \frac{\partial^2 V'}{\partial x' \partial y'} + \ldots, & x \frac{\partial^2 V'}{\partial x' \partial y'} + y \frac{\partial^2 V'}{\partial y'^2} + \ldots, & -, & -, \\ \qquad \frac{\partial U'}{\partial x'} \qquad , & \qquad \frac{\partial U'}{\partial y'} \qquad , & -, & -, \\ \qquad \frac{\partial V'}{\partial x'} \qquad , & \qquad \frac{\partial V'}{\partial y'} \qquad , & -, & -, \end{array} \right| = 0.$$

This determinant and its derivatives vanish for the point P', therefore the locus is a quadratic three-way cone with its vertex at P'. This cone is intersected by the tangent plane at P' in a pair of straight lines which is the required locus. If a point $x, y, \ldots$, be taken on either of these lines, we have three independent equations just sufficient to determine the ratios of the four differentials; i. e., just sufficient to determine the consecutive point P'', so that the tangent plane at this consecutive point will intersect the tangent plane at P' in the point selected. That these two consecutive tangent planes have no further intersection may be further shown by forming the equation of the plane that goes through their common intersection and through both the points P' and P''. The equations of this plane are

$$\Delta'' V' \, . \, \Delta U' - \Delta'' U' \, . \, \Delta V' = 0,$$
$$\Delta' V'' \, . \, \Delta U'' - \Delta' U'' \, . \, \Delta V'' = 0.$$

These equations in general represent a definite plane so long as P' and P'' are not coincident.

It would be of interest to examine the motion of the point of intersection along these lines as the point P'' circles about the point P', and to see whether at any time the consecutive tangent planes intersect in one of these lines.

These lines are not inflexional tangents to the surface; lines meeting the surface in three consecutive points do not generally exist in a space of more than three ways. For such lines would have to satisfy both

$$\Delta U' = 0,$$
$$\Delta V' = 0,$$
$$\cdots\cdots$$

and

$$\Delta^2 U' = 0,$$
$$\Delta^2 V' = 0,$$
$$\cdots\cdots$$

These equations, however, in general have only the point P' counted a multiple number of times in common. In general, then, in a space of more than three ways a surface is so twisted that there are no lines that meet the surface three times at a given point. This proof is easily extended to a surface in a space of more than four ways.

d. *The spreads that arise by considering the junctions of the consecutive tangent flats.*

Consider now any surface in n-fold space. Draw the 2-fold infinite system of tangent planes. Pass a four-flat through every two consecutive planes and there is a 3-fold infinite system of four-flats, forming in general a seven-spread. Each four-flat is met by the infinity of consecutive four-flats in the same plane. We may pass six-flats through every two consecutive four-flats. There is a 4-fold infinite system of six-flats constituting a ten-spread. This system of ruled loci in no wise resembles the system of developables we derived from a curve. Starting with a surface we cannot derive a system of developables in the same manner as when we start with a curve. The same is true if we start with any p-spread where $2 \overline{\overline{<}} p$. Only in case the p-spread lies in a $(p+1)$-flat do consecutive tangent p-flats intersect generally in $(p-1)$-flats; the only exception is in the case the p-spread is a curve.

II. Loci derived from an $(n-2)$-flat whose Equation involves a Single Arbitrary Parameter.

7. *Description of the loci.*

Let us consider next the system of loci represented by an $(n-2)$-flat whose equations involve a single arbitrary parameter. The parameter may enter rationally or irrationally. If it enters rationally we suppose it to enter to as high a degree as $\frac{n}{2}$ in each equation. Let the equations of the flat be

$$A = 0, \quad B = 0.$$

In these equations we suppose further that the linear function of the coördinates that appear as coefficients of the various powers of the parameter cannot be expressed in terms of fewer than $n+1$ linear functions of the coördinates. Eliminate the parameter from these equations and

we derive the equation of an $(n-1)$-spread S_{n-1}, which is ruled by the system of $(n-2)$-flats, F_{n-2}'s.*

Two consecutive F_{n-2}'s intersect in an $(n-4)$-flat, whose equations are,

$$A=0, \frac{\partial A}{\partial \lambda}=0, \; B=0, \frac{\partial B}{\partial \lambda}=0.$$

The elimination of the parameter from these equations gives a restricted system equivalent to three independent equations. The locus is an $(n-3)$-spread ruled by the F_{n-4}'s. S_{n-3} is a double spread on S_{n-1}.

Three consecutive F_{n-2}'s intersect in an $(n-6)$-flat F_{n-6}, whose equations are,

$$A=0, \frac{\partial A}{\partial \lambda}=0, \; \frac{\partial^2 A}{\partial \lambda^2}=0,$$

$$B=0, \frac{\partial B}{\partial \lambda}=0, \; \frac{\partial^2 B}{\partial^2 \lambda}=0.$$

If we eliminate the parameter from these equations we derive a restricted system equivalent to five independent equations. The locus is an $(n-5)$-spread S_{n-5}, ruled by the F_{n-6}'s. S_{n-5} is a triple spread on S_{n-1} and a double spread on S_{n-3}.

Similarly r consecutive F_{n-2}'s intersect in an $(n-2r)$-flat F_{n-2r}, whose equations are,

$$A=0, \frac{\partial A}{\partial \lambda}=0, \ldots, \frac{\partial^{r-1} A}{\partial \lambda^{r-1}}=0,$$

$$B=0, \frac{\partial B}{\partial \lambda}=0, \ldots, \frac{\partial^{r-1} B}{\partial \lambda^{r-1}}=0.$$

On the elimination of the parameter we derive a restricted system equivalent to $2r-1$ independent equations. The locus is an $(n-2r+1)$-spread, S_{n-2r+1}, ruled by the F_{n-2r}'s. S_{n-2r+1} is an r-tuple spread on S_{n-1}; it is a multiple spread on other spreads of the system.

Two distinct cases arise according as n is odd or even. If n is odd, then $\frac{n-1}{2}$ consecutive F_{n-2}'s intersect in a line, F_1, whose equations are,

* From now on we shall use S_k to denote the k-spread of this system.

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \ldots \frac{\partial^{\frac{n-3}{2}} A}{\partial \lambda^{\frac{n-3}{2}}} = 0,$$

$$B = 0, \frac{\partial B}{\partial \lambda} = 0, \ldots \frac{\partial^{\frac{n-3}{2}} B}{\partial \lambda^{\frac{n-3}{2}}} = 0.$$

If we eliminate the parameter from these equations we derive a restricted system equivalent to $n - 2$ independent equations. The locus is a surface S_2 ruled by the F_1's; it is an $\left(\frac{n-1}{2}\right)$-tuple surface on S_{n-1}.

Consecutive F_1's do not in general intersect for the $n + 1$ equations

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \ldots \frac{\partial^{\frac{n-1}{2}} A}{\partial \lambda^{\frac{n-1}{2}}} = 0,$$

$$B = 0, \frac{\partial B}{\partial \lambda} = 0, \ldots \frac{\partial^{\frac{n-1}{2}} B}{\partial \lambda^{\frac{n-1}{2}}} = 0,$$

have not in general any common solutions. If we regard these $n + 1$ equations as homogeneous in the $n + 1$ coördinates and form their resultant, the values of the parameter that cause it to vanish will give points where consecutive lines meet. The equations of these points may be formed by eliminating the parameter from the $n + 1$ equations, which gives a restricted system equivalent to n independent equations. These points are double points on S_2 and $\left(\frac{n+1}{2}\right)$-tuple points on S_{n-1}.

If n is even then $\frac{n}{2}$ consecutive F_{n-2}'s intersect in a point F_0, whose equations are,

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \ldots, \frac{\partial^{\frac{n-2}{2}} A}{\partial \lambda^{\frac{n-2}{2}}} = 0,$$

$$B = 0, \frac{\partial B}{\partial \lambda} = 0, \ldots, \frac{\partial^{\frac{n-2}{2}} B}{\partial \lambda^{\frac{n-2}{2}}} = 0.$$

The elimination of the parameter from these equations gives a restricted system equivalent to $n - 1$ independent equations. The locus is a curve

S_1, which is an $\left(\frac{n}{2}\right)$-tuple curve on S_{n-1}. There are not in general stationary points on S_1, for the $n+2$ equations

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \ldots, \frac{\partial^{\frac{n+1}{2}} A}{\partial \lambda^{\frac{n+1}{2}}} = 0,$$

$$B = 0, \frac{\partial B}{\partial \lambda} = 0, \ldots, \frac{\partial^{\frac{n+1}{2}} B}{\partial \lambda^{\frac{n+1}{2}}} = 0,$$

have not in general any common solutions at all.

If the equation of the $(n-2)$-flat involve k parameters connected by $k-1$ equations, the properties of the derived system of loci is the same as in the case just discussed.

8. *Mutual relations of the derived loci.*

Two consecutive F_{n-2}'s intersect in an F_{n-4}, three in an F_{n-6}, r in an F_{n-2r}, $\frac{n-1}{2}$ in an F_1, if n is odd, or $\frac{n}{2}$ in an F_0 if n is even. There is a 1-fold infinite system of each kind of flats. The F_{n-2}'s are generators of S_{n-1}, the F_{n-4}'s of S_{n-3}, the F_{n-2r}'s of S_{n-2r+1}. Let us consider the case where n is odd. Through any F_{n-4} pass two consecutive F_{n-2}'s, through any F_{n-2r} pass r consecutive F_{n-2}'s, through any F_1 pass $\frac{n-1}{2}$ consecutive F_{n-2}'s. Any F_{n-2} contains two consecutive F_{n-4}'s, three consecutive F_{n-6}'s, $\frac{n-1}{2}$ consecutive F_1's. Any F_{n-2r} contains two consecutive $F_{n-2(r+1)}$'s, any two consecutive F_{n-2r}'s determine one $F_{n-2(r-1)}$'s. We may then reverse the process and start with S_2, which lies in the space of n ways but in no flat space of a less number of ways. Through each two consecutive F_1's of this surface pass three-flats F_3's, these F_3's will generate a four-spread S_{n-4}. Through each two consecutive F_3's pass five-flats; this can be done as the F_{n-3}'s intersect consecutively in F_1's. These five-flats will generate a six-spread S_6. Finally, through each two consecutive F_{n-4}'s pass F_{n-2}'s; these F_{n-2}'s generate an $(n-1)$-spread S_{n-1}. If we start with the system of $(n-2)$-flats we come down finally to the surface, or starting with the surface we may work back to the system of $(n-2)$-flats.

If n is even, through any F_{n-4} pass two consecutive F_{n-2}'s, through any F_{n-2r} pass r consecutive F_{n-2}'s, through any F_0 pass $\frac{n}{2}$ consecutive F_{n-2}'s.

Any F_{n-2} contains two consecutive F_{n-4}'s, three consecutive F_{n-6}'s, $\frac{n}{2}$ consecutive F_0's. Any F_{n-2r} contains two consecutive $F_{n-2(r+1)}$'s and any two consecutive F_{n-2r}'s determine one $F_{n-2(r-1)}$ except in the case that $r = \frac{n}{2}$. We cannot then start with a curve and retrace our steps; two consecutive points of the curve S_1 do not determine uniquely a plane of the system. The F_2's of the system in general intersect consecutively in the points of S_1. Starting with such a system of planes we may retrace our steps. Through any two consecutive planes of the S_3 we may pass a four-flat. These four-flats are generators of S_5. Through any two consecutive F_4's we may pass six-flats; they are the generators of S_7. Finally through any two consecutive F_{n-4}'s pass $(n-2)$-flats; they are generators of S_{n-1}. We may retrace our steps only in case we do not begin with S_1.

9. *Director curves of the ruled* (n — 1)-*spread.*

Let the equation of such a ruled $(n-1)$-spread S_{n-1} be

$$\phi = 0.$$

We shall show that the equations of the generating flats of the spread may be represented by linear equations involving a single parameter. The equation in homogeneous coördinate of an arbitrary $(n-2)$-flat in n-fold space may be written

$$x = a_1 z + \beta_1 s + \ldots . + \gamma_1 w$$
$$y = a_2 z + \beta_2 s + \ldots . + \gamma_2 w.$$

In this form the equations of the $(n-2)$-flat, which we may call the $(n-2)$-flat AB, involve $2(n-1)$ independent arbitrary parameters. These parameters must be connected by $2(n-1)-1$ equation to make AB a generator of such an $(n-1)$-spread. We wish to connect these parameters in such a way that AB will be a generator of the S_{n-1} in question. The equations of a curve on ϕ are

$$\phi = 0,\ U_1 = 0,\ U_2 = 0, \ldots\ U_{n-2} = 0.$$

If we eliminate the coördinates between these equations and the equations of AB we derive a single equation in the $2(n-1)$ parameters. This resulting equation is the necessary and sufficient condition for AB to meet the curve. In a similar way we may derive $2(n-1)-1$ such conditions and make AB meet $2(n-1)-1$ curves on ϕ. If from these $2(n-1)-1$ equations and the equations of AB we elimi-

nate the parameters, we derive a single equation in the variables alone. It is the locus of all the $(n-2)$-flats that can be drawn to meet the curves in question, and so it necessarily includes all the generating flats of ϕ. It includes possibly other flats besides the generators of ϕ, but in this case the general locus will break up into several components, and one component is ϕ. This is the case in three-fold space.

The spreads $U_1, U_2, \ldots U_{n-2}$ may in each case be taken to be flats; then the director curves are plane curves. These are the director curves of ϕ; any or all of these curves may be plane, or they may be twisted to any extent permitted by the space. Any $2n-3$ curves in n-fold space may be taken as the director curves of a ruled $(n-1)$-spread. In three-fold space any three curves plane or twisted may be taken as the director curves of a ruled surface. In four-fold space, any five curves plane or twisted may be taken as the director curves of a ruled three-spread. In this case the generating planes intersect consecutively in the points of a sixth curve; so in four-fold space any five curves completely determine a sixth. In five-fold space seven curves plane or twisted may be taken as the director curves of a four-spread ruled by three flats. In six-fold space nine curves determine a five-spread ruled by four-flats. Consecutive four-flats intersect in planes and these in turn intersect consecutively in points. So in six-fold space nine curves determine a tenth.

10. *Multiple loci on the ruled* (n — 1)*-spread.*

Any generator of the $(n-1)$-spread is an $(n-2)$-flat F_{n-2}; it is met by any other generating F_{n-2} in an $(n-4)$-flat. If then $4 \overline{\overline{<}} n$ every generator is met by every other generator. If $n = 3$, any generator is met by only $m-2$ other generators, m being the order of the surface.*

For $4 \overline{\overline{<}} n$, any F_{n-2} contains a single infinity of $(n-4)$-flats where it is met by the other F_{n-2}'s. These are evidently double flats on S_{n-1}. On S_{n-1} there are in general a 2-fold infinite system of such $(n-4)$-flats constituting a double $(n-2)$-spread, Σ_{n-2} on S_{n-1}. In general, then, any $(n-1)$-spread S_{n-1} ruled by $(n-2)$-flats F_{n-2}'s has on it such a double $(n-2)$-spread Σ_{n-2} ruled by the 2-fold infinite system of $(n-4)$-flats. Σ_{n-4} has on it all those $(n-4)$-flats, F_{n-4}'s that arise from the intersection of consecutive F_{n-2}'s. These F_{n-4}'s generate S_{n-3}, which therefore lies on Σ_{n-2} and forms but an infinitesimal part of it.

Any three F_{n-2}'s intersect in an $(n-6)$-flat; there are in general a 3-fold infinite system of such $(n-6)$-flats constituting an $(n-3)$-

* Salmon, Geometry of Three Dimensions, p. 427.

spread Σ_{n-3}, a triple spread on S_{n-1}. S_{n-5} lies on Σ_{n-3}, and constitutes but an infinitesimal part of it. If n is sufficiently great there is a quadruple $(n-4)$-spread Σ_{n-4} ruled by the 4-fold infinite system of $(n-8)$-flats arising from the intersections of four F_{n-2}'s. S_{n-7} lies on S_{n-5}. We can go on in this manner until we reach a limit due to the narrowness of the space. If n is odd we have finally an $\left(\frac{n-1}{2}\right)$-tuple $\left(\frac{n+1}{2}\right)$-spread ruled by the $\left(\frac{n-1}{2}\right)$-fold infinite system of lines that arise from the intersection of $\frac{n-1}{2}$ generating F_{n-2}'s. There may be further an $\left(\frac{n+1}{2}\right)$-tuple $\left(\frac{n-1}{2}\right)$-spread made up of the $\left(\frac{n-1}{2}\right)$-fold infinite system of points that are the intersection of $\frac{n+1}{2}$ generating F_{n-2}'s, an $\left(\frac{n+3}{2}\right)$-tuple $\left(\frac{n-3}{2}\right)$-spread made up of the $\frac{n-3}{2}$-fold infinite system of points that are the intersections of $\frac{n+3}{2}$ generating F_{n-2}'s, etc., but these spreads do not always occur. In special cases the Σ_{n-2}, or some component of it, may be of greater multiplicity than $\frac{n+1}{2}$. In three-fold space a ruled surface generally has on it a double curve. This double curve, or some component of it, may, however, be of greater multiplicity than two. It is to be observed that S_{n-3} lies on Σ_{n-2}. In three-fold space this means that consecutive generators of a ruled surface, if they intersect at all, must intersect in points of the double curve. If n is even we have finally an $\left(\frac{n}{2}\right)$-tuple $\left(\frac{n}{2}\right)$-spread $\Sigma_{\frac{n}{2}}$ that is made up of the $\left(\frac{n}{2}\right)$-fold infinite system of points that arise from the intersection of $\frac{n}{2}$ generating F_{n-2}'s. There may be an $\left(\frac{n}{2}+1\right)$-tuple $\left(\frac{n}{2}-1\right)$-spread $\Sigma_{\frac{n}{2}-1}$ whose points are points of intersection of $\frac{n}{2}+1$ generating F_{n-2}'s, an $\left(\frac{n}{2}+2\right)$-tuple $\left(\frac{n}{2}-2\right)$-spread $\Sigma_{\frac{n}{2}-2}$ whose points are points of intersection of $\frac{n}{2}+2$ generating F_{n-2}'s, etc., though these spreads may not always be present.

11. *Special case where the parameter enters rationally.*

Let us consider the special case where the parameter enters rationally. Let the equation of the generating $(n-2)$-flat F_{n-2} be

$$A \equiv a\,t^l + b\,t^{l-1} + c\,t^{l-2} + \ldots . = 0,$$

$$B \equiv a'\,t^m + b'\,t^{m-1} + c'\,t^{m-2} + \ldots . = 0,$$

where $a, b, c, \ldots, a', b', c', \ldots$, are linear functions of the coördinates that cannot be expressed linearly in terms of fewer than $n+1$ linear functions of the coördinates. If we eliminate the parameter from these equations, we have the equation of the S_{n-1} ruled by the F_{n-2}'s; it is of order $l+m$. It is more convenient in what follows to use two parameters, λ and μ, that enter homogeneously into the equations.

Two consecutive generators intersect in the F_{n-4} whose equations are

$$\frac{\partial A}{\partial \lambda} = 0, \frac{\partial A}{\partial \mu} = 0, \frac{\partial B}{\partial \lambda} = 0, \frac{\partial B}{\partial \mu} = 0.$$

The elimination of the parameter from these equations gives a restricted system equivalent to three independent equations the locus is S_{n-3}, whose order is

$$2\,(l-1) + 2\,(m-1) = 2\,(l+m-2).$$

The order is found by expressing the conditions that the four equations have a common root. The locus of the intersections of three consecutive F_{n-2}'s is a locus of F_{n-6}'s; the equations of this locus are found by eliminating the parameters from the equations,

$$\frac{\partial^2 A}{\partial \lambda^2} = 0, \frac{\partial^2 A}{\partial \lambda\, \partial \mu} = 0, \frac{\partial^2 A}{\partial \mu^2} = 0,$$

$$\frac{\partial^2 B}{\partial \lambda^2} = 0, \frac{\partial^2 B}{\partial \lambda\, \partial \mu} = 0, \frac{\partial^2 B}{\partial \mu^2} = 0.$$

This gives a restricted system equivalent to five independent equations; it represents S_{n-5}, whose order is $3\,(l+m-4)$.

The r-tuple spread S_{n-2r+1} on S_{n-1} is represented by the equations that result from eliminating the parameters from the equations,

$$\frac{\partial^{r-1} A}{\partial \lambda^{r-1}} = 0, \frac{\partial^{r-1} A}{\partial \lambda^{r-2}\, \partial \mu} = 0, \ldots, \frac{\partial^{r-1} A}{\partial \mu^{r-1}} = 0,$$

$$\frac{\partial^{r-1} B}{\partial \lambda^{r-1}} = 0, \frac{\partial^{r-1} B}{\partial \lambda^{r-2}\, \partial \mu} = 0, \ldots, \frac{\partial^{r-1} B}{\partial \mu^{r-1}} = 0.$$

The equations then are of S_{n-2r+1} form a restricted system equivalent to $2r-1$ independent equations whose order is $r(l+m-2r+2)$. As we have seen, there are two cases according as n is odd or even. If n is odd we come down finally to an $\left(\frac{n-1}{2}\right)$-tuple surface S_2. The equations of S_2 are found by eliminating the parameters from the equations

$$\frac{\partial^{\frac{n-3}{2}} A}{\partial \lambda^{\frac{n-3}{2}}} = 0, \frac{\partial^{\frac{n-3}{2}} A}{\partial \lambda^{\frac{n-5}{2}} \partial \mu} = 0, \ldots, \frac{\partial^{\frac{n-3}{2}} A}{\partial \mu^{\frac{n-3}{2}}} = 0,$$

$$\frac{\partial^{\frac{n-3}{2}} B}{\partial \lambda^{\frac{n-3}{2}}} = 0, \frac{\partial^{\frac{n-3}{2}} B}{\partial \lambda^{\frac{n-5}{2}} \partial \mu} = 0, \ldots, \frac{\partial^{\frac{n-3}{2}} B}{\partial \mu^{\frac{n-3}{2}}} = 0.$$

The equations of S_2 form a restricted system equivalent to $n-2$ independent equations, whose order is $\frac{n-1}{2}(l+m-n+3)$.

There are also $\left(\frac{n+1}{2}\right)$-tuple points F_0's on S_{n-1}, though in general $\frac{n+1}{2}$ consecutive F_{n-2}'s do not intersect. If we form the resultant of the $n+1$ equations

$$\frac{\partial^{\frac{n-1}{2}} A}{\partial \lambda^{\frac{n-1}{2}}} = 0, \frac{\partial^{\frac{n-1}{2}} A}{\partial \lambda^{\frac{n-3}{2}} \partial \mu} = 0, \ldots, \frac{\partial^{\frac{n-1}{2}} A}{\partial \mu^{\frac{n-1}{2}}} = 0,$$

$$\frac{\partial^{\frac{n-1}{2}} B}{\partial \lambda^{\frac{n-1}{2}}} = 0, \frac{\partial^{\frac{n-1}{2}} B}{\partial \lambda^{\frac{n-3}{2}} \partial \mu} = 0, \ldots, \frac{\partial^{\frac{n-1}{2}} B}{\partial \mu^{\frac{n-1}{2}}} = 0,$$

we have a determinant of the $(n+1)$-st order, in which the parameters enter to the degree $\frac{n+1}{2}(l+m-n+1)$. There are then $\frac{n+1}{2}(l+m-n+1)$ values of the parameters that cause this determinant to vanish, and so this is the number of points F_0. We can find the equations of these points by eliminating the parameters from these $n+1$ equations. The result is a restricted system equivalent to n independent equations. The order of the system is $\frac{n+1}{2}(l+m-n+1)$. This is another proof of the number of points F_0 on S_{n-1}.

In case n is even we have finally the $\binom{n}{2}$-tuple curve whose equations are found by eliminating the parameters from the equations,

$$\frac{\partial^{\frac{n-2}{2}} A}{\partial \lambda^{\frac{n-2}{2}}} = 0, \frac{\partial^{\frac{n-2}{2}} A}{\partial \lambda^{\frac{n-4}{2}} \partial \mu} = 0, \ldots, \frac{\partial^{\frac{n-2}{2}} A}{\partial \mu^{\frac{n-2}{2}}} = 0,$$

$$\frac{\partial^{\frac{n-2}{2}} B}{\partial \lambda^{\frac{n-2}{2}}} = 0, \frac{\partial^{\frac{n-2}{2}} B}{\partial \lambda^{\frac{n-4}{2}} \partial \mu} = 0, \ldots, \frac{\partial^{\frac{n-2}{2}} B}{\partial \mu^{\frac{n-2}{2}}} = 0.$$

The order of the restricted system is $\frac{n}{2}(l + m - n + 2)$, the order of S_1.

We find the equation of the double spread Σ_{n-2} on S_{n-1}, by imposing on the equations of the generating F_{n-2} the conditions that they have two common roots in the parameter. These conditions are,*

$$\text{(II)} \qquad \left\| \begin{matrix} a, & b, & c, \ldots\ldots \\ & a, & b, \ldots\ldots \\ & & \ldots\ldots\ldots \\ a', & b', & c', \ldots\ldots \\ & a', & b', \ldots\ldots \\ & & \ldots\ldots\ldots \end{matrix} \right\| = 0,$$

where there are $l + m - 2$ rows and $l + m - 1$ columns. This is a restricted system equivalent to two independent equations; the order of the system is $\frac{1}{2}(l + m - 1)(l + m - 2)$. On Σ_{n-2} must be S_{n-3}. We find the equations of Σ_{n-3} by expressing the conditions that the equations of the generating flat have three common roots in the parameter.† The result is a restricted system similar in form to (II), in which, however, there are only $l + m - 4$ rows and $l + m - 2$ columns. This restricted system is equivalent to three independent equations, and its order is $\frac{1}{6}(l + m - 2)(l + m - 3)(l + m - 4)$.

The equations of Σ_{n-r} are found by expressing the conditions that the equations of the generating $(n - r)$-flat have r roots in common. By an extension of the previous method we derive a restricted system of the same form as (II), in which, however, there are only $l + m - 2(r - 1)$ rows and $l + m - (r - 1)$ columns. This is a restricted system equiva-

* Salmon, Higher Algebra, Art. 275. †Ibid., Art. 285.

lent to r independent equations, the order of the system is $\frac{1}{r!}(l + m - r + 1)(l + m - r) \ldots . (l + m - 2r + 2)$.

Whether n is odd or even we have finally a curve Σ_1 of multiplicity $n - 1$, whose equations are found by expressing the conditions that the equations of the generating $(n - 2)$-flat have $n - 1$ roots in the parameter in common. We derive a restricted system of the same form as (II) in which however there are $l + m - 2(n - 2)$ rows and $l + m - (n - 2)$ columns. The order of this system is $\frac{1}{(n-1)!}(l + m - n + 2)(l + m - n + 1) \ldots . (l + m - 2n + 4)$. This curve has n-tuple points on it whose equations are found by expressing the conditions that the equations of the generating $(n - 2)$-flat have n roots in common. We again have a restricted system of the same form as (II), in which, however, there are $l + m - 2(n - 1)$ rows and $l + m - n + 1$ columns. The order of this system is $\frac{1}{n!}(l + m - n + 1)(l + m - n) \ldots . (l + m . 2n + 2)$, which is the number of points in question. For $n = 3$ these formulae for the order agree with those given in Salmon.*

A very special case is where the parameter enters only linearly in one of the equations of the generating $(n - 2)$-flat. Let the equations of the flat be

$$A \equiv a t + b = 0,$$

$$B \equiv a' t^m + b' t^{m-1} + \ldots . = 0,$$

where we make the same suppositions regarding $a, b, a', b', \ldots$, as before. The S_{n-1} in this case is a ruled spread with m sheets through the $(n - 2)$-flat, whose equations are

$$a = 0, b = 0;$$

it has no other multiple locus on it at all. Consecutive generating F_{n-2}'s of the system intersect in the flat, whose equations are,

$$a = 0, b = 0, B = 0, \frac{\partial B}{\partial t} = 0.$$

All the F_{n-4}'s of the system lie in the same $(n - 2)$-flat; they generate a developable $(n - 3)$-spread S'_{n-3} in this flat. S'_{n-3} is the section by this flat of the developable $(n - 1)$-spread enveloped by the $(n - 1)$-flat B. Consecutive generating F_{n-2}'s of S_{n-1} intersect in generating F_{n-4}'s of

* Salmon, Geometry of Three Dimensions, p. 428.

S'_{n-3}. By means of an $(n-3)$-way developable lying in an $(n-2)$-flat and two arbitrary curves we can generate a ruled $(n-1)$-spread by taking all the $(n-2)$-flats that can be drawn through the enveloping $(n-3)$-flats of the developable so as to meet both curves.

We have seen that the section of an $(n-1)$-way developable by an $(n-1)$-flat gave an $(n-2)$-way developable of the same nature, so here the section of an $(n-1)$-spread ruled by $(n-2)$-flats by an $(n-1)$-flat gives an $(n-2)$-spread of the same nature as the $(n-1)$-spread.

III. Loci derived from an $(n-k)$-flat whose Equations involve a Single Arbitrary Parameter.

12. *Description of the derived loci.*

We shall complete the general theory by considering the locus of the 1-fold infinite system of $(n-k)$-flats, where $2 \overline{\overline{<}} k$ whose equations all contain a single arbitrary parameter. Let the k equations of the flat be

$$A = 0, B = 0, \ldots, C = 0.$$

The equations of the locus of these F_{n-k}'s are found by eliminating the parameter from these equations. The result is a restricted system equivalent to $k-1$ independent equations.

The locus is an $(n-k+1)$-spread S_{n-k+1} ruled by the F_{n-k}'s. Any two consecutive F_{n-k}'s intersect in an $(n-2k)$-flat F_{n-2k} whose equations are

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, B = 0, \frac{\partial B}{\partial \lambda} = 0, \ldots.$$

If we eliminate the parameter from these equations, we derive a restricted system equivalent to $2k-1$ independent equations. The locus is an $(n-2k+1)$-spread S_{n-2k+1} ruled by the F_{n-2k}'s; it is a double spread on S_{n-k}.

Any three consecutive F_{n-2k}'s intersect in an $(n-3k)$-flat F_{n-3k} whose equations are,

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \frac{\partial^2 A}{\partial \lambda^2} = 0, B = 0, \frac{\partial B}{\partial \lambda} = 0, \ldots$$

The elimination of the parameter from these equations gives a restricted system equivalent to $3k-1$ independent equations. Their locus is an $(n-3k+1)$-spread ruled by the F_{n-3k}'s. S_{n-3k+1} is a triple spread on S_{n-k+1}.

The equations of the locus of the intersections of r consecutive F_{n-k}'s are found by eliminating the parameter from the equations

$$A = 0, \frac{\partial A}{\partial \lambda} = 0, \ldots \frac{\partial^{r-1} A}{\partial \lambda^{r-1}} = 0,$$

$$B = 0, \frac{\partial B}{\partial \lambda} = 0, \ldots \ldots \ldots \ldots$$

This gives a restricted system equivalent to $rk - 1$ independent equations. The locus is an $(n - rk + 1)$-spread ruled by the F_{n-rk}'s, it is an r-tuple spread on S_{n-k+1}.

There are k cases according as $n \equiv 0 \pmod{k}$, $n \equiv 1 \pmod{k}$, ..., $n \equiv k - 1 \pmod{k}$. In the first case we come finally to a curve S_1 which is an $\left(\frac{n}{k}\right)$-tuple curve on S_{n-k+1}. In the second case we come down finally to a system of lines F_1's which are generators of a ruled surface S_2. In the last case we come down finally to a k-spread ruled by $(k-1)$-flats. There are on S_k in general special points where two consecutive F_{k-1}'s intersect.

13. *Multiple loci on the spread; mutual relations of the system of spreads.*

S_{n-k+1} has on it in general multiple loci that arise from the intersection of non-consecutive F_{n-k}'s. Any F_{n-k} intersects every other F_{n-k} in an $(n - 2k)$-flat; there is in general a 2-fold infinite system of such $(n - 2k)$-flats constituting a double $(n - 2k + 2)$-spread Σ_{n-2k+2} on S_{n-k+1}. Evidently S_{n-2k+1} lies on Σ_{n-2k+2}. Any three F_{n-k}'s intersect in an $(n - 3k)$-flat; there is a 3-fold infinite system of such $(n - 3k)$-flats, they constitute in general a triple $(n - 3k + 3)$-spread Σ_{n-3k+3} on S_{n-k+1}. S_{n-3k+1} lies on Σ_{n-3k+3}. Any r consecutive F_{n-k}'s intersect in an $(n - rk)$-flat; there is an r-fold infinite system of such $(n - rk)$-flats in general, constituting an r-tuple $(n - rk + r)$-spread Σ_{n-rk+r} on S_{n-k+1}, on which lies S_{n-rk+1}.

Finally the locus of the intersection of any α F_{n-k}'s where α is the greatest integer in $\frac{n}{k}$ is an α-tuple $[n - \alpha(k-1)]$-spread $\Sigma_{n-\alpha(k-1)}$ on S_{n-k+1}; it is ruled by the α-fold infinite system of $(n - \alpha k)$-flats.

The question arises, When, in general, do these double loci cease to exist? The double spread is in general an $(n - 2k + 2)$-spread Σ_{n-2k+2}. To have a continuous locus of double points we must generally have

$$n - 2k + 2 \gtreqless 1 \text{ or } k \lesseqgtr \frac{n+1}{2}.$$

For values of k that satisfy this condition there is in general a continuous locus of double points. If

$$n - 2k + 2 = 0, \text{ or } k = \frac{n+2}{2}$$

there is in general only a finite number of double points on the locus. If

$$n - 2k + 2 < 0, \text{ or } k > \frac{n+2}{2}$$

there are in general no double points on the locus.

If there enter into the equations of the generating $(n-k)$-flat ρ parameters connected by $\rho - 1$ equations the properties of the system of related loci will be similar to those of the system just described.

Any two consecutive F_{n-k}'s intersect in an F_{n-2k} while through any F_{n-2k} pass two consecutive F_{n-k}'s. Any three consecutive F_{n-k}'s intersect in an F_{n-3k} while through any F_{n-3k} pass two consecutive F_{n-2k}'s and three consecutive F_{n-k}'s. Any two consecutive F_{n-rk}'s determine in general one $F_{n-k(r-1)}$. An exception may occur if $r = a$ the greatest integer in $\frac{n}{k}$. Thus, if $n \equiv 0$ (mod k), two consecutive points of S_1 do not determine a $(k+1)$-flat where $2 \lesseqgtr k$.

If $n \equiv 1$ (mod k), two consecutive lines of S_2 do not determine a $(k+1)$-flat, except in the case $k = 2$. In the last case, however, where $n \equiv k - 1$ (mod k), two non-intersecting $(k-1)$-flats do determine a $(2k-1)$-flat. Only in this last case can we retrace the steps if we come down to the last spread. We can always retrace the steps if we do not come down to this last case.

14. *Director spreads of the ruled spread.*

The equation in homogeneous coördinates of any $(n-k)$-flat, $2 \lesseqgtr k$, may be written

$$x = \alpha_1 s + \beta_1 t + \ldots . + \gamma_1 w,$$
$$y = \alpha_2 s + \beta_2 t + \ldots . + \gamma_2 w,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$z = \alpha_k s + \beta_k t + \ldots . + \gamma_k w.$$

In this form the equations of the flat contain $k(n-k+1)$ independent parameters. These parameters must be connected by $k(n-k+1) - 1$ equations for this $(n-k)$-flat to be a generator of such a ruled $(n-k+1)$-spread. Any curve is given by the equations

$$\phi = 0,$$
$$\chi = 0,$$
$$\dots\dots$$

a restricted system equivalent to $n - 1$ independent equations. If we eliminate the coördinates between the equations of the flat and curve, we derive a restricted system equivalent to $k - 1$ independent equations in the parameters alone. These are the conditions that must be satisfied for the $(n - k)$-flat to meet the curve. In a similar way we may derive a restricted system equivalent to $k - p$ independent equations in the parameters alone which are the necessary and sufficient conditions for the $(n - k)$-flat to meet a certain p-spread where $1 \overline{\overline{<}} p \overline{\overline{<}} k - 1$. We may have then curves, surfaces, . . . , or p-spreads where $1 \overline{\overline{<}} p \overline{\overline{<}} k - 1$ for the director loci of a ruled $(n - k + 1)$-spread. The numbers of loci of each kind that must be taken are λ, μ, . . . ν, ρ, namely, non-negative integers chosen to satisfy the equation

$$\lambda\,(k - 1) + \mu\,(k - 2) + \dots + \nu\,.\,2 + \rho\,.\,1 = k\,(n - k + 1) - 1.$$

If we consider a group of one or more points as a director locus of the spread, we have to select integers to satisfy

$$\kappa\,.\,k + \lambda\,(k - 1) + \dots + \rho\,.\,1 = k\,(n - k + 1) - 1.$$

We may apply this to special cases. The director loci of a ruled surface in three-fold space are three curves. We may take one curve and a group of κ points, in which case the ruled surface is reducible and has for its components κ cones whose vertices are the κ points and whose common base is the curve in question. In four-fold space the director loci of a ruled surface may be five surfaces, three surfaces and one curve, or one surface and two curves. The ruled surface in each case consisting of all the lines that can be drawn to meet all the director loci. In the same space the director loci of a three-spread ruled by planes may be taken to be five curves.

If the director loci be all taken on any S_{n-k+1}, then the locus of all the $(n - k)$-flats that can be drawn to meet these director loci will include as one of its components the S_{n-k+1} in question; it may or may not have other components.

There are several special cases illustrative of these methods that can be worked out in still greater detail. Some of these I hope to make the subject of another paper.